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# Gravitation of the Klein–Gordon Scalar Field

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This paper presents an exact solution of the Einstein–Klein–Gordon equations in the static and spherically symmetric case and points out the differences between it and Yilmaz's solution. In addition, the essential difference between the exact solution and the post-Newtonian approximate solution is also shown.

## 1. INTRODUCTION

Even though the Einstein field equations are highly nonlinear partial differential equations, there are numerous exact solutions to them (Kramer *et al.*, 1980). These solutions include the well-known spherically symmetric solutions of Schwarzschild, Reissner, Tolman, and Friedmann, and the axisymmetric solutions of Weyl, Kerr, and Einstein–Rosen, which are of great importance in physical applications (Carmeli, 1982). However, the analysis of the Einstein–Klein–Gordon (hereafter EKG) equations (the Einstein field equations with a nonvanishing energy-momentum tensor which arise from the scalar field) has not been put forward as a physical problem, although Yilmaz (1958, 1972) obtained some exact solutions in his new gravitation theory in which the field equations were similar to the EKG equations. Yilmaz's results were not recognized by orthodox theorists, mainly because of his uncommon theory. In the present paper, I consider the EKG equations according to the work of Reissner and Nördstrom on a spherically symmetric charged body. In Section 2, an exact solution for a zero-mass scalar field is obtained in the static and spherically symmetric case; the space-time of this solution is asymptotically flat. In addition, an approximate solution according to the post-Newtonian method in this case is also shown. In Section 3, the physical meaning of

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Yilmaz's solution is discussed and the differences between his result and mine are demonstrated.

## 2. BASIC EQUATIONS AND SOLUTION

Neglecting the cosmological constant, the Einstein equation is generally known as

$$G_\mu^\nu = \kappa T_\mu^\nu \tag{1}$$

where $G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R$ is the mixed Einstein tensor, $\kappa = 8\pi G/c^4$ is Einstein's gravitational constant, and $T_\mu^\nu$ is the energy-momentum tensor.

In the presence of gravitation, the Klein–Gordon equation takes the form

$$\Box\Phi + m^2\Phi = 0 \tag{2}$$

where we choose the natural units $G = \hbar = c = 1$; $\Box$ and $m$ denote the D'Alembertian operator and the mass of the scalar particle, respectively. The corresponding mixed energy-momentum tensor of the scalar field is given by (Davis, 1972)

$$T_\mu^\nu = g^{\nu\sigma}\frac{\partial\Phi}{\partial x^\sigma}\frac{\partial\Phi}{\partial x^\mu} - \frac{1}{2}\delta_\mu^\nu\left(g^{\rho\sigma}\frac{\partial\Phi}{\partial x^\rho}\frac{\partial\Phi}{\partial x^\sigma} - m^2\Phi^2\right) \tag{3}$$

where $\delta_\mu^\nu$ denotes the Kronecker delta. Equations (1)–(3) make up the EKG equations. In the simplest spherically symmetric case, the most general metric has the form

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = e^\alpha\,dt^2 - e^\beta\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{4}$$

where $\alpha$ and $\beta$ are functions of the coordinates $r$ and $t$.

Considering the static condition, the EKG equations are decomposed into the following nonvanishing components:

$$G_0^0 = -e^{-\beta}\left(\frac{1}{r^2} - \frac{\beta'}{r}\right) + \frac{1}{r^2} = \kappa T_0^0 = \frac{\kappa}{2}(m^2\Phi^2 + e^{-\beta}\Phi'^2) \tag{5}$$

$$G_1^1 = -e^{-\beta}\left(\frac{1}{r^2} + \frac{\alpha'}{r}\right) + \frac{1}{r^2} = \kappa T_1^1 = \frac{\kappa}{2}(m^2\Phi^2 - e^{-\beta}\Phi'^2) \tag{6}$$

$$G_2^2 = -\tfrac{1}{2}e^{-\beta}\left(\alpha'' + \frac{\alpha'^2}{2} + \frac{\alpha' - \beta'}{r} - \frac{\alpha'\beta'}{2}\right) = \kappa T_2^2 = \kappa T_0^0 \tag{7}$$

$$G_3^3 = G_2^2 \tag{8}$$

$$\Phi'' + \left(\frac{2}{r} + \frac{\alpha' - \beta'}{2}\right)\Phi' - m^2 e^\beta\Phi = 0 \tag{9}$$

where the prime denotes differentiation by $r$.

Using $G_0^0 \pm G_1^1 = \kappa(T_0^0 \pm T_1^1)$, we can reduce (5)–(8) to

$$e^{-\beta}\left(\frac{\beta' - \alpha'}{r} - \frac{2}{r^2}\right) + \frac{2}{r^2} = \kappa m^2 \Phi^2 \tag{10}$$

$$\frac{\beta' + \alpha'}{r} = \kappa \Phi'^2 \tag{11}$$

Equations (9)–(11) are three independent equations of our problem and equation (7) can be deduced from them.

If the mass of the scalar particle is equal to zero and the fields $\Phi$ and $\Phi'$ are not, equations (9)–(11) can be simplified to

$$\frac{\Phi'''}{\Phi'} - \frac{2\Phi''^2}{\Phi'^2} - \frac{2}{r}\frac{\Phi''}{\Phi'} - \frac{2}{r^2} = \frac{\kappa}{2}(r\Phi'\Phi'' + \Phi'^2) \tag{12}$$

$$e^{\beta} = -1 - r\frac{\Phi''}{\Phi'} \tag{13}$$

$$e^{\alpha} = \frac{g^2}{\Phi'^2 r^4} e^{\beta} \tag{14}$$

where $g$ is an integral constant.

Taking a variable quantity transformation $X = r\Phi'$, we find that equation (12) becomes

$$\frac{X''}{X} - \frac{2X'^2}{X^2} - \frac{\kappa X X'}{2r} = 0 \tag{15}$$

It is clear that $X' = 0$ is a particular solution of the above equation; however, it is not reasonable in physics because the corresponding $e^{\alpha}$ and $e^{\beta}$ are equal to zero.

A cumbersome effort yields another exact solution for the nonlinear equation (15):

$$X = \frac{c_1}{(c_2 r^2 + \kappa c_1^2/2)^{1/2}} \tag{16}$$

where $c_1$, $c_2$ are constant. Hence the corresponding scalar field is

$$\Phi' = \frac{g}{r(r^2 + \kappa g^2/2)^{1/2}} \tag{17}$$

$$\Phi = \Phi_0 - \left(\frac{2}{\kappa}\right)^{1/2} a \sinh\left[\left(\frac{\kappa}{2}\right)^{1/2} \frac{g}{r}\right] \tag{18}$$

and the metric is given by

$$e^{\alpha} = \frac{g^2 c_1}{c_2^2} = 1 \tag{19}$$

$$e^{\beta} = \frac{r^2}{r^2 + \kappa g^2/2} \tag{20}$$

where the mixed constant term is chosen to be unity, as the space-time takes the Minkowski form when $r \to \infty$. Obviously, the metric shows that the space-time is asymptotically flat and the solution has no Schwarzschild term $1/r$. Substituting the above solutions into the basic equations (7), (9), (10), and (11), we can check their rightness; the answer is positive. For comparing the exact solution and the approximate solution, we sòlve the EKG equations according to the post-Newtonian method (Will, 1981). First, we extract the energy-momentum tensor by solving the KG equation in the Minkowski space-time, then substitute it into the Einstein equations to find the first-order modification to the Minkowski metric.

In the static and spherically symmetric Minkowski space-time, the KG equation is

$$\Phi'' + \frac{2}{r}\Phi' - m^2\Phi = 0 \tag{21}$$

This is a spherical Bessel equation with zero and imaginary quantity $im$. The solution is the linear combination of $[\sin(imr)]/imr$ and $-[\cos(imr)]/imr$. With a suitable choice of coefficients, $\Phi$ becomes the Yukawa potential:

$$\Phi = g\frac{e^{-mr}}{r} \tag{22}$$

where $g$ denotes the strong interaction constant; I choose the same character as in equation (14) because there is a possible relation between the two constants (see the following). The corresponding energy-momentum tensor is given by

$$T_0^0 = T_2^2 = T_3^3 = -\frac{1}{2}\Phi'^2 + \frac{1}{2}m^2\Phi^2 \tag{23}$$

$$T_1^1 = \frac{1}{2}\Phi'^2 + \frac{1}{2}m^2\Phi^2 \tag{24}$$

Substituting equations (23) and (24) into equation (1), we obtain the first-order metric modification (the process of analysis is exact and has no approximation):

$$e^{-\beta} = 1 - \frac{2M}{r} + \kappa(1 + mr)\frac{g^2 e^{-2mr}}{2r^2} \tag{25}$$

$$\alpha' = -\frac{1}{r} + \left[\frac{1}{r} + \kappa\left(m + \frac{1}{2r}\right)\frac{g^2 e^{-2mr}}{r^2}\right]e^{\beta} \tag{26}$$

where $M$ is an integral constant and can be defined as the mass of a gravitating body at the original point. When $m = 0$ and $g \neq 0$, the integral is, for $M^2 > \kappa g^2/2$,

$$e^{\alpha} = \left(\frac{r - M - (M^2 - \kappa g^2/2)^{1/2}}{r - M + (M^2 - \kappa g^2/2)^{1/2}}\right)^{M/(M^2 - \kappa g^2/2)^{1/2}} \tag{27}$$

for $M^2 < \kappa g^2/2$,

$$\alpha = \frac{2M}{(\kappa g^2/2 - M^2)^{1/2}} \operatorname{arctg} \frac{r - M}{(\kappa g^2/2 - M^2)^{1/2}} \tag{28}$$

and for $M^2 = \kappa g^2/2$,

$$\alpha = -\frac{2M}{r - M} \tag{29}$$

$$e^{-\beta} = 1 - \frac{2M}{r} + \frac{\kappa g^2}{2r^2} \tag{30}$$

The metric shows a Schwarzschild term $2M/r$ and is the Schwarzschild metric when $m$ and $g$ are equal to zero, and interestingly, when $m$ and $M$ are equal to zero the integral of equation (24) leads to $e^{\alpha} = 1$ and the metric is the exact solution we obtain, which is why the integral constant in equation (14) is denoted by the strong interaction constant $g$. The essential difference between the exact solution and the approximate one raises the question of the rightness of the post-Newtonian method, which has been widely used in the weak gravitational field.

## 3. SCALAR FIELD AND DISCUSSION

The coupling interaction of the scalar field and gravitation appears in the form of EKG equations according to the traditional Einstein theory; a similar form exists also in some uncommon theories of gravitation, such as Yilmaz's new gravitational theory, Hoyle's creation field theory (Narlikar, 1985), and Yu's (1989) geometric creation theory. The general form of the equations of these theories can be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} + \alpha(\Phi_\mu \Phi_\nu - \frac{1}{2} g_{\mu\nu} \Phi_\alpha \Phi^\alpha) + g(\Phi) \tag{31}$$

$$\Box\Phi = f(\Phi) \tag{32}$$

where $a$ is a constant (negative for creation field, $\kappa$ for EKG theory), and $g(\Phi)$ and $f(\Phi)$ are functions of the scalar field. In the case of $f(\Phi) = 0$, $g(\Phi) = 0$, and $T_{\mu\nu} = 0$, the above equations are different from the EKG equations only by a constant coefficient. Yilmaz solved an equation of EKG form in the static and spherically symmetric case and found an exact solution (an isotropic metric):

$$ds^2 = e^{-2m/\rho}\, dt^2 - e^{2m/\rho}[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{33}$$

$$\Phi = \pm i \left(\frac{2}{a}\right)^{1/2} \frac{m}{\rho} \tag{34}$$

where $i$ denotes the complex operator, and $m$ is explained by Yilmaz as the total mass producing the gravitational field (hence $m$ is a real positive number and scalar curvature has no singularity at $\rho = 0$); here we consider it as a constant of the scalar field (hence $m$ may be an imaginary number).

We can see that the metric is approximately the Schwarzschild form when $m/r \ll 1$. After checking its rightness in isotropic coordinates, we perform a coordinate transformation by using a superfunction:

$$\rho e^{m/\rho} = r \tag{35}$$

Then the line element takes the form of equation (2). Equation (15) must contain this solution, and gives

$$m^2 + \frac{\kappa g^2}{2} = 0 \tag{36}$$

The related metric is

$$e^\alpha = e^{-2m/\rho} \tag{37}$$

$$e^\beta = \frac{\rho^2}{\rho^2 - m^2} \tag{38}$$

$m$ must be an imaginary number because $\kappa$ and $g$ are real positive numbers; hence the space-time of Yilmaz's solution is oscillatory. The two solutions are definitely not equal, but the two metrics are equal at a series of quantum $r$ values

$$r = im/n\pi = (\kappa/2)^{1/2} g/n\pi \tag{39}$$

where $n$ is a positive integer. It seems that the exact solution we obtain is a flat one corresponding to Yilmaz's solution, just as the Minkowski solution is a flat one corresponding to the Schwarzschild solution. The difference of the two exact solutions can be also shown by comparing the scalar curvatures given by them. The scalar curvature of the isotropic metric is $(2m^2/\rho^4)e^{-2m/\rho}$, and the other is $-\kappa g^2/r^4$ or $(2m^2/\rho^4)e^{-4m/\rho}$. Only one of the scalar curvatures has a singularity at $r = 0$ and the two metrics cannot be made equal by any coordinate transformation.

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